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Obstructions and lines of marginal stability from the world-sheet

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Abstract

The behaviour of supersymmetric D-branes under deformations of the closed string background is studied using world-sheet methods. We explain how lines of marginal stability and obstructions arise from this point of view. We also show why $\mathcal{N}=2$ B-type branes may be obstructed against (cc) perturbations, but why such obstructions do not occur for $\mathcal{N}=4$ superconformal branes at $c=6$, *i.e.* for half-supersymmetric D-branes on K3. Our analysis is based on a field theory approach in superspace, as well as on techniques from perturbed conformal field theory.

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1 Introduction

The moduli space of supersymmetric D-branes typically depends on the closed string background of the setup, and its dimension and structure may change drastically as one varies the background. One interesting effect that may happen is that a supersymmetric D-brane may cease to be supersymmetric upon a small closed string perturbation that preserves the supersymmetry of the closed string. In this case one usually says that there is an *obstruction* for adjusting the D-brane to the closed string deformation. Another even more drastic effect is that the D-brane may decay into a superposition of D-branes; this is what happens as one crosses a *line of marginal stability*. Both phenomena occur for supersymmetric D-branes in type II compactifications on Calabi-Yau manifolds. In this case, the combined bulk-boundary moduli space has a space-time interpretation in terms of F-terms and D-terms of the low-energy theory. F-term constraints are related to preserving an extended world-sheet supersymmetry (and hence to the absence of obstructions), whereas the requirement of conformal invariance and charge integrality (that controls the lines of marginal stability) is encoded in the D-terms.

In this paper we study these phenomena from a world-sheet point of view for theories with different amounts of supersymmetry. In particular, we shall concentrate on two cases: theories with $\mathcal{N} = 1$ spacetime supersymmetry in four dimensions, in which case the corresponding world-sheet theory is an $\mathcal{N} = 2$ superconformal field theory [1], and theories with $\mathcal{N} = 1$ spacetime supersymmetry in six dimensions (which corresponds to $\mathcal{N} = 2$ spacetime supersymmetry in four dimensions), in which case the world-sheet theory has an $\mathcal{N} = 4$ superconformal symmetry with $c = 6$ [2]. In each case we shall give a world-sheet interpretation of obstructions and lines of marginal stability. One of our main results is that for D-branes that preserve the $\mathcal{N} = 4$ superconformal symmetry with $c = 6$ — this is in particular the case for half-supersymmetric branes on K3 — obstructions cannot occur.

There are two natural and successful approaches to the analysis of world-sheet theories. First one may regard them as 2-dimensional supersymmetric field theories using superspace techniques, but without using the conformal symmetry. In this formulation the supersymmetry variation of the closed string perturbation is proportional to the integral of a total derivative. On world-sheets without boundary these perturbations thus preserve supersymmetry, but in the presence of D-branes (*i.e.* world-sheets with boundaries) the total derivatives may lead to non-trivial boundary terms. Such non-trivial boundary terms can sometimes be cancelled against the supersymmetry variation of a correction term involving boundary fields [3]. However, the boundary theory may not contain a suitable boundary field, so that such a resolution is impossible. If this is the case, the corresponding D-brane is obstructed against the closed string perturbation. As we shall explain, for world-sheet theories with $\mathcal{N} = 2$ supersymmetry, B-type branes may be obstructed in this way against chiral perturbations, but not against twisted chiral perturbations. On the other hand, there are no such obstructions for world-sheet theories with $\mathcal{N} = 4$ supersymmetry.

This analysis can capture obstructions, but it does not detect the potential presence of lines of marginal stability. In order to study the latter one also has to take into account the conformal symmetry; thus one is led to employ techniques of perturbed conformal field theory. In this approach deformations of the closed string background correspond

to perturbations of the conformal field theory by exactly marginal bulk fields. In the situation with $\mathcal{N} = 2$ superconformal world-sheet symmetry, the corresponding bulk fields are either (cc), (ca), (ac) or (aa) fields [4, 5]. These fields are always exactly marginal on world-sheets without boundary. In the presence of boundaries, however, they may cease to be exactly marginal. If this is the case they induce an RG flow on the boundary that drives the brane to a configuration that is compatible with the deformed closed string background [6]; the bulk theory itself remains unaffected unless one also includes the backreaction of the brane [7]. A typical example of such an RG flow is what happens when one crosses a line of marginal stability in the moduli space.

At a generic point of the moduli space, far away from lines of marginal stability, no such RG flow is induced, and thus the boundary condition remains conformal. Nevertheless the bulk perturbation will generically modify the gluing conditions of the symmetry algebra, including those that describe how the supercharges are to be identified at the boundary. For $\mathcal{N} = 2$ superconformal B-type branes it may happen that a (cc) perturbation breaks the $\mathcal{N} = 2$ superconformal symmetry at the boundary, *i.e.* that the brane is obstructed. On the other hand, if the bulk and boundary theory are $\mathcal{N} = 4$ superconformal with $c = 6$, then we can show that the change in the gluing condition for the $\mathcal{N} = 2$ (or $\mathcal{N} = 4$ generators) under a (cc) perturbation can always be absorbed into a redefinition of the gluing conditions that identify the left- and right-moving supercharges. Thus $\mathcal{N} = 4$ superconformal branes are not obstructed under such perturbations.

The paper is organised as follows. In section 2 we analyse the problem from a field theory point of view. In particular, in section 2.1 we use an $\mathcal{N} = 2$ superspace formulation to explain how obstructions may arise for $\mathcal{N} = (2, 2)$ supersymmetric theories. Section 2.2 then deals with the case of $\mathcal{N} = (4, 4)$ supersymmetry. The conformal field theory approach is described in section 3. Section 3.2 and 3.3 deal with the case of $\mathcal{N} = 2$ B-type branes under (cc) and (ac) perturbations, respectively. The analysis for the case with $\mathcal{N} = 4$ superconformal symmetry is described in section 3.4. We illustrate our findings in section 4 with two examples, a simple brane configuration on a T^4 torus, as well as a D-brane on K3 at the orbifold point $T^4/\mathbb{Z}_4$. Finally, section 5 contains our conclusions. There are two appendices where some of the more technical material is described: appendix A gives a detailed account of our superspace conventions, in particular of the harmonic superspace that is used in the $\mathcal{N} = (4, 4)$ case. In appendix B we discuss constraints on the fusion rules of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ chiral primary operators.

2 The superspace analysis

In this section we analyse the problem of brane obstructions under deformations by bulk moduli, formulating the theory in terms of an action in superspace. In a first step we will review the situation with $\mathcal{N} = (2, 2)$ supersymmetry, using an $\mathcal{N} = (2, 2)$ superspace formulation, see [8, 9]. In section 2.2 we shall then consider the situation with $\mathcal{N} = (4, 4)$ supersymmetry.

2.1 The $\mathcal{N} = 2$ analysis

Let us begin by describing the two-dimensional world-sheet theory in standard $\mathcal{N} = (2, 2)$ superspace

$$\mathbb{R}^{(1,1|2,2)} = \mathbb{R}_L^{(1|2)} \times \mathbb{R}_R^{(1|2)} = \{x^+, \theta^+, \bar{\theta}^+\} \times \{x^-, \theta^-, \bar{\theta}^-\} \ , \tag{2.1}$$

where the two factors represent the two light-cone sectors. Our conventions follow [9], and are described in appendix A.1. We parametrise an arbitrary supervariation as

$$\delta = \epsilon_+ \mathcal{Q}_- - \epsilon_- \mathcal{Q}_+ - \bar{\epsilon}_+ \bar{\mathcal{Q}}_- + \bar{\epsilon}_- \bar{\mathcal{Q}}_+ \ . \tag{2.2}$$

Here the signs are chosen so that δ is a hermitian operator. In order to study the supervariation in the presence of a boundary we need to specify which linear combinations of the supercharges are preserved at the boundary. We shall always take the boundary to be along the line $x^+ = x^-$, and we shall always consider B-type boundary conditions

$$\epsilon \equiv \epsilon_+ = -\epsilon_- \ , \qquad \bar{\epsilon} \equiv \bar{\epsilon}_+ = -\bar{\epsilon}_- \ . \tag{2.3}$$

For this boundary condition we shall then study chiral and twisted chiral deformations (that correspond to (cc) and (ac) deformations, respectively). Note that this then covers already the general case since mirror symmetry exchanges A-type[1] and B-type boundary conditions, as well as chiral and twisted chiral perturbations.

It will turn out that in both cases supervariations will only close up to total derivatives, leading to boundary contributions which must be cancelled by introducing new terms involving fields on the boundary. For twisted chiral perturbations this can always be achieved by using bulk fields that are taken to the boundary. For chiral perturbations, on the other hand, we will in general need bona fide boundary fields that do not come from the bulk. It is however not a priori clear that the boundary spectrum contains the appropriate fields; if it does not, then we cannot cancel the boundary contribution of the supersymmetry variation, and the brane will be obstructed.

2.1.1 Chiral deformations

Suppose now that we consider the variation of the (bulk) action by a chiral superfield, *i.e.* by the term

$$\Delta S = \int d^2x \, d\theta^- d\theta^+ \ \Phi|_{\bar{\theta}^\pm = 0} \ . \tag{2.4}$$

From the conformal field theory point of view to be described below, this corresponds to a (cc) deformation. If we write out the superfield in components as in (A.4), then ΔS equals

$$\Delta S = \int d^2x \, F(x^\pm) \ . \tag{2.5}$$

We are interested in the supersymmetry variation described by δ of this perturbation. To this end we calculate

$$\delta \, \Delta S = \int d^2x \, d\theta^- d\theta^+ \left(\epsilon_+ \mathcal{Q}_- - \epsilon_- \mathcal{Q}_+ - \bar{\epsilon}_+ \bar{\mathcal{Q}}_- + \bar{\epsilon}_- \bar{\mathcal{Q}}_+ \right) \Phi \ . \tag{2.6}$$

[1] In distinction to (2.3) A-type boundary conditions are given by $\epsilon \equiv \epsilon_+ = \bar{\epsilon}_-$ and $\bar{\epsilon} \equiv \bar{\epsilon}_+ = \epsilon_-$.

It is easy to see that the terms involving $\mathcal{Q}_\pm$ act trivially. To evaluate the other two terms we use that $\bar{\mathcal{D}}_\pm\Phi = 0$, and thus find

$$\begin{aligned}\delta\,\Delta S &= \int d^2x\, d\theta^- d\theta^+ \left(2i\bar\epsilon_+\theta^-\partial_-\Phi - 2i\bar\epsilon_-\theta^+\partial_+\Phi\right) \\ &= -2i\int d^2x\ \left(\bar\epsilon_-\partial_+\psi_- + \bar\epsilon_+\partial_-\psi_+\right)\ .\end{aligned} \tag{2.7}$$

The last term is a total derivative and thus vanishes on a world-sheet without boundary. In the presence of a boundary along $x^1 = 0$, the ∂_1 derivative gives the contribution

$$\delta\Delta S = -i\int_{x^1=0} dx^0\ \left(\bar\epsilon_-\psi_- - \bar\epsilon_+\psi_+\right)\ . \tag{2.8}$$

The full perturbation we are interested in contains also the complex conjugate of (2.4), namely

$$\overline{\Delta S} = \int d^2x\, d\bar\theta^+ d\bar\theta^-\ \bar\Phi\big|_{\theta^\pm=0}\ , \tag{2.9}$$

where $\bar\Phi$ is the complex conjugate of Φ, and thus defines an anti-chiral superfield. By a similar calculation to the above (using the expansion of the anti-chiral superfield $\bar\Phi$ as in (A.6)) one finds that

$$\begin{aligned}\delta\overline{\Delta S} &= 2i\int d^2x\, d\bar\theta^+ d\bar\theta^- \left(\epsilon_+\bar\theta^-\bar\theta^+\partial_-\bar\psi_+ - \epsilon_-\bar\theta^+\bar\theta^-\partial_+\bar\psi_-\right) \\ &= i\int_{x^1=0} dx^0\ \left(\epsilon_-\bar\psi_- - \epsilon_+\bar\psi_+\right)\ .\end{aligned} \tag{2.10}$$

Altogether the supersymmetry variation of the total chiral deformation equals

$$\delta(\Delta S + \overline{\Delta S}) = i\int_{x^1=0} dx^0\ \left(\bar\epsilon_+\psi_+ - \bar\epsilon_-\psi_- + \epsilon_-\bar\psi_- - \epsilon_+\bar\psi_+\right)\ . \tag{2.11}$$

Using the explicit form of the B-type boundary conditions (2.3) this can be rewritten as

$$\begin{aligned}\delta(\Delta S + \overline{\Delta S}) &= i\int_{x^1=0} dx^0\ \left(-\epsilon(\bar\psi_+ + \bar\psi_-) + \bar\epsilon(\psi_+ + \psi_-)\right) \\ &= -i\int_{x^1=0} dx^0 d\bar\theta\,\epsilon\,\bar\Phi + i\int_{x^1=0} dx^0 d\theta\,\bar\epsilon\,\Phi\ ,\end{aligned} \tag{2.12}$$

where we used the B-type conditions to define the boundary Grassmann variables $\theta = \theta^+ + \theta^-$ and $\bar\theta = \bar\theta^+ + \bar\theta^-$. In general, this term cannot be cancelled by a boundary term involving only bulk fields. Thus B-type branes may be obstructed under (cc) deformations. From a space-time point of view, this reflects the fact that the (cc) fields enter the superpotential W, and allowed motions in parameter space are constrained to the zero locus of ∂W. Note also that the (cc) fields as well as the B-type boundary conditions survive the topological B-twist; thus the (holomorphic) superpotential W is calculable in the topological B-model.

Sometimes it is possible to preserve the full $\mathcal{N} = 2$ by introducing additional boundary degrees of freedom with appropriate supersymmetry transformations. This has, in particular, been done in the context of Landau Ginzburg models [10, 11] (for a review

see [12, 13]). In some situations, however, even this is not enough to preserve $\mathcal{N} = 2$ supersymmetry, as can be seen for example in [14, 15]. While the $\mathcal{N} = 2$ supersymmetry may thus be broken by the deformation, it is always possible to cancel the variation of the $\mathcal{N} = 1$ supersymmetry variation that corresponds to $\epsilon = \bar{\epsilon}$. In fact, since the $\mathcal{N} = 1$ supersymmetry is a gauge symmetry in string theory, this is crucial for the consistency of the perturbation. The boundary term that cancels the $\mathcal{N} = 1$ variation is simply given by

$$\mathcal{B} = -i \int_{x^1=0} dx^0 (\phi - \bar{\phi}) \ , \tag{2.13}$$

whose variation yields

$$\delta\mathcal{B} = -i \int_{x^1=0} dx^0 \ \big(\epsilon(\psi_+ + \psi_-) - \bar{\epsilon}(\bar{\psi}_+ + \bar{\psi}_-)\big) \ . \tag{2.14}$$

This indeed cancels (2.12) for $\epsilon = \bar{\epsilon}$.

2.1.2 Twisted chiral deformations

The other possibility is that we modify the action by a twisted chiral perturbation (corresponding to an (ac) field)

$$\Delta S = \int d^2x \, d\bar{\theta}^- d\theta^+ \ U|_{\bar{\theta}^+=\theta^-=0} \ , \tag{2.15}$$

where U is a twisted chiral supermultiplet. Since the calculation of the supersymmetry variation proceeds much like in the case of the twisted perturbation, we will omit the details. The resulting boundary term is

$$\delta\Delta S = i \int_{x^1=0} dx^0 \, (\epsilon_+\chi_+ - \bar{\epsilon}_-\bar{\chi}_-) \ . \tag{2.16}$$

Combining this with its complex conjugate, one obtains

$$\delta(\Delta S + \overline{\Delta S}) = i \int_{x^1=0} dx^0 \, (\epsilon_+\chi_+ - \bar{\epsilon}_-\bar{\chi}_- - \bar{\epsilon}_+\bar{\chi}_+ + \epsilon_-\chi_-) \ . \tag{2.17}$$

This boundary term can always be cancelled by adding the boundary integral

$$\mathcal{B} = -i \int_{x^1=0} dx^0 (v - \bar{v}) \ , \tag{2.18}$$

where v is the lowest component of the twisted chiral superfield, while its complex conjugate $\bar{v}$ is the lowest component of the twisted anti-chiral superfield (see appendix A.1). Indeed, the variation of (2.18) cancels precisely (2.17). This ties in with the expectation that B-type branes are never obstructed against (ac) deformations [16]. The complete story is however more complicated since there are in general lines of marginal stability in the (ac) moduli space along which D-branes will decay [17, 18, 19]. We will return to this phenomenon in section 3.

2.2 Theories with $\mathcal{N} = 4$

We now wish to generalise the results of the previous section to cases where $\mathcal{N} = (2,2)$ supersymmetry is enhanced to $\mathcal{N} = (4,4)$. We shall discuss the problem using an $\mathcal{N} = 4$ superspace formulation. Different $\mathcal{N} = 4$ superspace formulations are known, in particular projective superspace [20, 21, 22, 23], harmonic superspace [24, 25], as well as conventional $\mathcal{N} = (4,4)$ superspace using constrained superfields [26]. In the following we shall employ harmonic superspace; some technical details can be found in appendix A.2 (see also [27, 28, 29, 30]).

2.2.1 $\mathcal{N} = 4$ harmonic superspace

The simplest way to enhance the superspace (2.1) to $\mathcal{N} = (4,4)$ is to promote all Grassmann coordinates to doublets under an $SU(2) \times SU(2)$ R-symmetry group[2]

$$\mathbb{R}^{(1,1|4,4)} = \mathbb{R}^{(1|4)}_L \times \mathbb{R}^{(1|4)}_R = \{x^+, \theta^a_i\} \times \{x^-, \theta^{\bar{a}}_{\bar{\imath}}\} \ , \tag{2.19}$$

where $i, \bar{\imath} = 1, 2$ and the index a and $\bar{a}$ distinguishes spinors from their conjugates (see (A.20)). The explicit construction of (off-shell) superfields on $\mathbb{R}^{(1,1|4,4)}$ is a rather delicate issue (see *e.g.* [26]), and we therefore prefer to formulate our theory on a slightly modified (harmonic) superspace. To this end we define

$$\mathbb{H}^{(1,1+4|4,4)} = \mathbb{R}^{(1,1|4,4)} \times \frac{SU(2)_L}{U(1)_L} \times \frac{SU(2)_R}{U(1)_R} = \{x^+, \theta^{a(\pm,0)}, u_i^{(\pm,0)}\} \times \{x^-, \theta^{\bar{a}(0,\pm)}, u_{\bar{\imath}}^{(0,\pm)}\} \ , \tag{2.20}$$

where we have introduced new variables

$$\{u_i^{(\pm,0)}\} \in \frac{SU(2)_L}{U(1)_L} \qquad \text{and} \qquad \{u_{\bar{\imath}}^{(0,\pm)}\} \in \frac{SU(2)_R}{U(1)_R} \ . \tag{2.21}$$

The $SU(2)$ properties of these variables become apparent if one writes them in matrix form like in (A.26). The essential idea is now to identify these groups with the left- and right-moving $SU(2)_f$, respectively.[3] This allows us to define new Grassmann variables via projection with the harmonic variables

$$\theta^{a(\pm,0)} = \theta^a_i u^{(\pm,0),i}, \qquad \text{and} \qquad \theta^{\bar{a}(0,\pm)} = \bar{\theta}^{\bar{a}}_{\bar{\imath}} u^{(0,\pm),\bar{\imath}}, \tag{2.22}$$

(for more details see appendix A.3). In these conventions, the analogue of the supervariation (2.2) takes the form

$$\delta = \epsilon^{a(-,0)} \mathcal{Q}_a^{(+,0)} + \epsilon^{a(+,0)} \mathcal{Q}_a^{(-,0)} + \epsilon^{\bar{a}(0,-)} \mathcal{Q}_{\bar{a}}^{(0,+)} + \epsilon^{\bar{a}(0,+)} \mathcal{Q}_{\bar{a}}^{(0,-)} \ . \tag{2.23}$$

[2]In the following we shall drop the superscripts $\pm$ of the Grassmann variables in order to avoid cluttering our formulae. Note that the Grassmann variables also transform as doublets with respect to another $SU(2)$. We shall call the former $SU(2)$ action (which acts on the indices i and $\bar{\imath}$) 'flavour' $SU(2)_f$, while the latter $SU(2)$ (rotating the indices a and $\bar{a}$) will be referred to as 'colour' $SU(2)_c$. This notation follows the nomenclature of [31].

[3]The actual construction of $\mathbb{H}^{(1,1+4|4,4)}$ as a coset space is a little bit more involved and will not be presented here since it is only of minor importance for the rest of this work. A detailed description of the construction can however, for example, be found in [30] (see also [32]).

We also need to formulate the $\mathcal{N}=4$ analogue of the chiral and twisted-chiral deformations of the $\mathcal{N}=2$ theory which we have discussed in section 2.2. In the $\mathcal{N}=4$ language, both deformations can be described in a unified manner as (see [24])

$$S_{\rm def} = \int d^2x \int du^{(\pm,0)} \int du^{(0,\pm)} \int d^2\theta^{(+,0)} \int d^2\theta^{(0,+)} \theta^{a(+,0)} \theta^{\bar{a}(0,+)} \zeta_{a\bar{a}} \Phi^{(+,+)} \ , \tag{2.24}$$

where $\zeta_{a\bar{a}}$ is a parameter which essentially picks out various components of the $\mathcal{N}=4$ superfield. Notice that upon focusing on a specific $\mathcal{N}=(2,2)$ subalgebra, these components can be precisely identified with the (cc), (ac), (ca) and (aa) perturbations which we have discussed in section 2.1. Note also that the integrand of this superspace integral contains explicitly the Grassmann variables, which naively indicates that the deformation does not preserve supersymmetry (even without boundaries). However, given the special properties of the superfields $\Phi^{(+,+)}$ and the fact that $\zeta_{a\bar{a}}$ is a u-independent constant, the above terms are actually supersymmetric. One way to see this is to rewrite (2.24) in component language. We first perform the Grassmann integration in (2.24) which yields

$$\begin{aligned} S_{\rm def} &= \int d^2x \int du^{(\pm,0)} \int du^{(0,\pm)} \int d^2\theta^{(+,0)} \int d^2\theta^{(0,+)} \theta^{a(+,0)} \theta^{\bar{a}(0,+)} \, \zeta_{a\bar{a}} \, \Phi^{(+,+)} \\ &= \int d^2x \int du^{(\pm,0)} \int du^{(0,\pm)} \int d\theta^{(+,0)}_a \int d\theta^{(0,+)}_{\bar{a}} \, \zeta_{a\bar{a}} \, \Phi^{(+,+)} \Big|_{\theta^{a(+,0)}=\theta^{\bar{a}(0,+)}=0} \\ &= \int d^2x \int du^{(\pm,0)} \int du^{(0,\pm)} \, \zeta_{a\bar{a}} \, F^{a\bar{a}}. \end{aligned} \tag{2.25}$$

Since the integrand is independent of the harmonic variables the $\int du^{(\pm,0)}$ and $\int du^{(0,\pm)}$ integrals are trivial, and the deformation can be written as

$$S_{\rm def} = \int d^2x \, \zeta_{a\bar{a}} \, F^{a\bar{a}} \ . \tag{2.26}$$

In particular, this term has the same structure as the $\mathcal{N}=2$ terms (2.5) or (2.15). Since the supervariaton of $F^{a\bar{a}}$ is a total derivative

$$\delta F^{a\bar{a}} = 2i \, \epsilon^a_i \, \partial_+ \psi^{i\bar{a}} + 2i \, \epsilon^{\bar{a}}_{\bar{\imath}} \, \partial_- \psi^{\bar{\imath}a} \, , \tag{2.27}$$

it is now obvious that (2.26) is in fact invariant under supersymmetry transformations (at least as long as there are no boundaries).

2.2.2 Boundary conditions and deformations

Before we can study the behaviour of boundary conditions under these bulk deformations we need to review the structure of the $\mathcal{N}=4$ preserving boundary conditions. The automorphism (R-symmetry) group of the $\mathcal{N}=4$ algebra is $SU(2)_c \times SU(2)_f \cong SO(4)$. Thus a general $\mathcal{N}=4$ preserving boundary condition will be labelled by an element in this group, where the corresponding automorphism describes how the left- and right-moving supercharges are identified at the boundary $x^+ = x^-$.

In the context of string theory, not all such boundary conditions are of interest. This comes from the fact that the overall $\mathcal{N}=(1,1)$ supersymmetry is a gauge symmetry in

string theory, and thus that the corresponding supercharges must be identified without any non-trivial automorphism [33]. Since we want to impose a fixed $\mathcal{N}=1$ symmetry, we restrict ourselves to identifications of the form

$$\mathcal{Q}^i_a = U^i{}_{\bar\jmath}\, \mathcal{Q}^{\bar\jmath}_a \ , \tag{2.28}$$

where $U^i{}_{\bar\jmath}$ is a given $SU(2)$ matrix. This corresponds to imposing the boundary conditions for the Grassmann variables

$$\theta^a_i = U_i{}^{\bar\jmath}\, \theta^a_{\bar\jmath} \ , \tag{2.29}$$

where $U_i{}^{\bar\jmath}$ is the inverse of the matrix in (2.28). We should also mention that from the harmonic point of view different choices of the matrix $U_i{}^{\bar\jmath}$ correspond to different parametrisations of the coset space (2.21), and therefore can also be interpreted as simple coordinate transformations in our fully covariant notation.

In the presence of such a boundary condition we are then interested in the supervariations for which

$$\epsilon^a_i = U_i{}^{\bar\jmath}\, \epsilon^a_{\bar\jmath} \ , \tag{2.30}$$

again with the same $U_i{}^{\bar\jmath}$. If we now perturb the theory by the bulk deformation (2.24), then the supervariation equals, using the component language (2.26)

$$\begin{aligned}\delta S_{\rm def} &= 2i \int d^2x\, \zeta_{a\bar a} \left(\epsilon^a_i \partial_+ \psi^{i\bar a} + \epsilon^{\bar a}_{\bar\imath} \partial_- \psi^{\bar\imath a}\right) \\ &= 2i \int dx^0 \zeta_{a\bar a} \left(\epsilon^a_i \psi^{i\bar a} - \epsilon^{\bar a}_{\bar\imath} \psi^{\bar\imath a}\right) = 2i \int dx^0\, \zeta_{[a\bar a]} \epsilon^a_i \psi^{i\bar a} \ .\end{aligned} \tag{2.31}$$

In the last step we have used that (2.29) implies that also ψ^{ia} and $\psi^{\bar\imath a}$ satisfy a similar relation at the boundary. Since $\zeta_{[a\bar a]}$ is an antisymmetric 2×2 matrix, we can always write it as

$$\zeta_{[a\bar a]} = \epsilon_{a\bar a} \left(\epsilon^{b\bar b} \zeta_{b\bar b}\right) \equiv \epsilon_{a\bar a}\, g \ . \tag{2.32}$$

It is then easy to see that the above supervariation is precisely cancelled by the supervariaton of the boundary term

$$S^{\rm bdy}_{\rm def} = \int dx^0\, g\, \varphi^{ij}\, \epsilon_{ij} \ . \tag{2.33}$$

This implies that on the level of the superspace analysis, $\mathcal{N}=4$ supersymmetric branes are never obstructed.

3 Conformal field theory analysis

In the previous section we showed, using a superspace approach on the world sheet that $\mathcal{N}=4$ supersymmetric branes are not obstructed against bulk perturbations that preserve the $\mathcal{N}=(4,4)$ supersymmetry in the bulk. Note that this did not deal with the question of the conformal symmetry of the theory, which is of course crucial *e.g.* in applications to string theory. In this section we thus address the same problem using conformal field theory methods. Again we begin by studying the analysis from an $\mathcal{N}=2$ point of view.

3.1 Generalities for $\mathcal{N}=2$ theories

We are interested in (closed string) perturbations that preserve the $\mathcal{N}=(2,2)$ superconformal symmetry in the bulk.[4] There are four different types of perturbations that have this property, and they are usually referred to as (cc), (aa), (ac) and (ca). In all cases, the perturbation is of the form

$$\int d^2z\, \widehat{\Phi}(z,\bar{z}) = \int d^2z \big(G_{-1/2}\tilde{G}_{-1/2}\Phi\big)(z,\bar{z}) \ , \qquad \text{where} \qquad \widehat{\Phi} = G_{-1/2}\tilde{G}_{-1/2}\Phi \ , \tag{3.1}$$

and G_r, $\tilde{G}_r$ are the modes of the left- and right-moving $\mathcal{N}=1$ supercurrent, which in terms of the supercurrents of the $\mathcal{N}=2$ algebra is simply $G_r = G^+_r + G^-_r$ and similarly for the right-movers (see appendix B.)[5] Depending on which case one considers Φ in (3.1) is a chiral (c) or anti-chiral (a) primary with respect to the left- and right-moving $\mathcal{N}=2$ algebra. So for example, in the (cc) case, we have

$$G^+_{-1/2}\Phi = \tilde{G}^+_{-1/2}\Phi = 0 \ , \qquad \text{so that} \qquad \widehat{\Phi} = G^-_{-1/2}\tilde{G}^-_{-1/2}\Phi \ . \tag{3.2}$$

This is the perturbation that corresponds to the 'chiral deformation' of section 2.1.1. As is well known [35], for chiral primaries the U(1) charge and the conformal dimension is related as $h=q/2$, while for anti-chiral primaries we have instead $h=-q/2$. For the marginal bulk perturbations we need that $h=\bar{h}=\frac{1}{2}$, and thus $q=\pm 1=\bar{q}$. In particular, $\widehat{\Phi}$ then always has vanishing U(1) charges.

We are interested in such perturbations in the presence of a boundary. As already mentioned before, there are two natural boundary conditions for $\mathcal{N}=(2,2)$ theories, namely A-type and B-type branes. In terms of the conformal field theory description, they are characterised by their gluing conditions, specifying how the left- and right-moving fields are identified at the boundary. If we take the boundary to be the real axis, the two cases are

$$T(z)=\tilde{T}(\bar{z}) \ , \quad G^\pm(z)=\tilde{G}^\mp(\bar{z}) \ , \quad J(z)=-\tilde{J}(\bar{z}) \ , \quad \text{for } z=\bar{z} \quad \text{(A-type)} \ , \tag{3.3}$$

$$T(z)=\tilde{T}(\bar{z}) \ , \quad G^\pm(z)=\tilde{G}^\pm(\bar{z}) \ , \quad J(z)=\tilde{J}(\bar{z}) \ , \quad \text{for } z=\bar{z} \quad \text{(B-type)} \ , \tag{3.4}$$

where the right-moving fields are denoted by a tilde. Because of mirror symmetry the behaviour of A-type D-branes under (cc) and (ac) deformations is the same as that of B-type D-branes under (ac) and (cc) deformations, respectively, and similarly for (ca) and (aa) perturbations. It is therefore sufficient to concentrate on the case of B-type boundary conditions. Furthermore, the analysis of (aa) deformations is essentially identical to that of a (cc) deformation, and similarly for (ca) and (ac). Thus we shall concentrate on B-type branes under (cc) and (ac) deformations.

In either case, there are two issues to consider. First, as discussed in [6] (see also [36, 37]), a bulk perturbation may break the conformal symmetry on the boundary and induce a non-trivial RG flow. This will be the case provided that the bulk perturbation $\widehat{\Phi}$

[4]In the context of an $\mathcal{N}=(4,4)$ theory, these perturbations will then also preserve the $\mathcal{N}=(4,4)$ symmetry in the bulk [34].

[5]Unlike in section 2, $\pm$ now denotes the U(1) charge of the operators, whereas right-movers are denoted by a tilde. Bar will be used later on for complex conjugation.

will switch on a marginal boundary field as it approaches the boundary. Then the brane will generically flow a finite distance to a different boundary condition, and we should not expect to be able to say much about the symmetries it preserves.

On the other hand, if no such marginal or relevant operator is switched on, then the brane will only adjust infinitesimally to an infinitesimal bulk perturbation. Then we can ask, following [36], to which extent the gluing condition will be modified.

In the following we shall analyse these questions separately for the (cc) and (ac) perturbations.

3.2 $\mathcal{N}=2$ B-type branes under (cc) deformations

Let us first ask whether the bulk perturbation by the (cc) field $\widehat{\Phi}$ will break the conformal invariance of the boundary theory. To this end we need to consider the bulk-boundary OPE as the field $\widehat{\Phi}$ approaches the boundary. By the usual doubling trick [38] we can replace the bulk field $\widehat{\Phi}$ by two chiral fields, one in the upper half-plane, and one in the lower half-plane. Since we are dealing with a B-type boundary condition, both of these fields are $G^-_{-1/2}$ descendants of chiral primary fields with $q=1$ and $h=\frac{1}{2}$, and we shall denote them by ϕ_c and $\bar{\phi}_c$, respectively.

The possible boundary fields that are switched on are constrained by the fusion rules of the $\mathcal{N}=2$ superconformal algebra [39, 40]. In particular, it follows from the analysis of appendix B.2 that the fusion of two chiral primary fields with $h=\frac{1}{2}$ and $q=1$ only contains the superconformal families

$$\phi_c \otimes \bar{\phi}_c = [\varphi_{h=1,q=2}] \oplus [G^+_{-1/2}\varphi^+_{q=1}] \ . \tag{3.5}$$

The first term is the usual chiral ring product and represents the so-called 'even fusion rules'; the second term describes the odd fusion that is possible in this case. Only the U(1) charge of the fields φ^+ is fixed to be $q=1$, but the conformal dimension of φ^+ is not constrained; in particular there may therefore be more than one such channel. However, for each of these fields the usual unitarity bound implies that $h \geq \frac{1}{2}$. Furthermore, $h=\frac{1}{2}$ is excluded, since φ^+ would then be a chiral primary field for which $G^+_{-1/2}\varphi^+ = 0$.

Since fusion rules describe the product structure of the corresponding superconformal families, we can then also conclude what the leading terms in the OPE of $G^-_{-1/2}\phi_c$ must be. In fact, from (3.5) and U(1) charge conservation it follows that

$$(G^-_{-1/2}\phi_c) \otimes (G^-_{-1/2}\bar{\phi}_c) = G^-_{-1/2}G^-_{-3/2}\varphi_{h=1,q=2} + G^-_{-1/2}\varphi^+_{q=1} + \text{higher descendants} \ . \tag{3.6}$$

Because the conformal dimension of φ^+ satisfies $h > \frac{1}{2}$, it then follows that none of the fields on the right hand side are marginal or relevant. This then implies that no marginal or relevant boundary field is switched on by the perturbation by $\widehat{\Phi}$. Thus the perturbation by the (cc) field $\widehat{\Phi}$ does not break the conformal symmetry.

3.2.1 Modifying the gluing condition

Since the bulk perturbation does not break the conformal invariance of the boundary condition, we can ask whether it will affect the superconformal gluing conditions, in particular those corresponding to $G^{\pm}$. To analyse this effect, we need to apply the analysis

of [36] to the present context. To first order in the bulk perturbation, the change in the gluing condition for $G^\pm$ is determined by (compare eq. (3.9) of [36] — we are using here that the automorphism ω is trivial for B-type branes)

$$\Delta G^\pm = \lambda \lim_{y\to 0} \int_{\mathbb{H}_+} d^2w \left(G^\pm(z) - \tilde{G}^\pm(\bar z) \right) \widehat{\Phi}(w,\bar w) \ , \tag{3.7}$$

in the limit where z approaches the boundary, *i.e.* $y = \operatorname{Im} z \to 0$. Using the doubling trick [38] we can think of this as a chiral correlator on the full plane, where $\tilde{G}^\pm(\bar z)$ is the usual chiral $G^\pm$ field at the image point, and we write $\widehat{\Phi}$ as a product of two chiral fields, namely $G^-_{-1/2}\phi_c$ at w, and $G^-_{-1/2}\bar\phi_c$ at $\bar w$. Then the above expression has four poles. For the case of the field G^+ each of them is of the form

$$\begin{aligned} G^+(z)\left(G^-_{-1/2}\phi_c\right)(w) &\simeq \frac{1}{(z-w)^2} V(G^+_{1/2}G^-_{-1/2}\phi_c, w) + \frac{1}{(z-w)} V(G^+_{-1/2}G^-_{-1/2}\phi_c, w) \\ &= \frac{2}{(z-w)^2}\, V(\phi_c, w) + \frac{2}{(z-w)}\, \partial_w V(\phi_c, w) \\ &= 2\,\frac{d}{dw}\Big(\frac{1}{z-w}\, V(\phi_c, w)\Big)\ . \end{aligned} \tag{3.8}$$

Here we are using the same notation as in [36], and $V(\phi, z)$ denotes the field corresponding to the state ϕ at the position z. On the other hand, for G^- there is no pole at all. Thus the gluing condition for G^- will not be modified. For G^+, on the other hand, using the fact that (3.8) is a total derivative, we can do the integral in (3.7) and find

$$\Delta G^+ = 2\lambda\pi \lim_{y\to 0}\Big[\phi_c(z)\left(G^-_{-1/2}\bar\phi_c\right)(\bar z) - \left(G^-_{-1/2}\phi_c\right)(z)\,\bar\phi_c(\bar z)\Big]\ . \tag{3.9}$$

In general this limit is non-trivial since the fusion rules (3.6) allow for non-trivial terms. In fact, since ϕ_c is a fermionic field, the linear combination that appears in (3.9) is just the $G^-_{-1/2}$ descendant of (3.5), and hence we have that

$$\Big[\phi_c(z)\left(G^-_{-1/2}\bar\phi_c\right)(\bar z) - \left(G^-_{-1/2}\phi_c\right)(z)\,\bar\phi_c(\bar z)\Big] = G^-_{-1/2}\,\varphi_{h=1,q=2} \ + \ G^-_{-1/2}G^+_{-1/2}\varphi^+_{q=1} \ + \ \text{desc.} \tag{3.10}$$

The limit $y \to 0$ is then well defined, and only the first term survives, thus giving

$$\Delta G^+ = 2\lambda\pi\, G^-_{-1/2}\,\varphi_{h=1,q=2}\ . \tag{3.11}$$

Hence the gluing condition for G^+ is modified unless $\varphi_{h=1,q=2}$ vanishes, whereas that for G^- is unmodified. Because of unitarity we also need to perturb by the conjugate (aa) field, which in turn only changes G^-, but not G^+. If the boundary condition just preserves the $\mathcal{N}=2$ superconformal algebra, its boundary spectrum does not contain a chiral field of conformal dimension $h=1$ and U(1) charge $q=2$. Thus the correction term $\Delta G^+ = G^-_{-1/2}\varphi_{h=1,q=2}$ is not part of the symmetry algebra of the boundary theory, and hence this modification cannot be absorbed into a redefinition of the gluing condition. In this case the bulk perturbation therefore breaks the $\mathcal{N}=2$ superconformal symmetry of the boundary condition. This means that the brane is obstructed against this (cc) deformation.

In the $\mathcal{N} = 2$ case, the condition for whether there is an obstruction is therefore whether the bulk boundary correlator

$$\langle \Phi_{cc}(z,\bar{z})\, \varphi_{h=1,q=2}(x) \rangle \tag{3.12}$$

vanishes or not. In fact, this correlator is a 'topological' quantity, which can be calculated in topological string theory. For the case of a 3-dimensional Calabi-Yau manifold, it is (after spectral flow) precisely equal to the bulk boundary correlator that was calculated in [41] using the Kapustin-Li formula [42]. As was explained in [36], this correlator vanishes if and only if the bulk field is BRST-exact when brought to the boundary, which in turn is the condition of [15] for the absence of an obstruction to first order. In the case of K3, spectral flow works slightly differently, and (3.12) becomes precisely the 'charge' of the bulk field. This controls the obstruction theory for K3, as was explained in [43].

3.2.2 Restoring the $\mathcal{N} = 1$ superconformal symmetry

The above analysis shows that the $\mathcal{N} = 2$ superconformal symmetry is broken by a (cc) perturbation if (3.12) does not vanish. If this is the case, also the $\mathcal{N} = 1$ superconformal symmetry is broken. From a string theory point of view this is *not acceptable* since the $\mathcal{N} = 1$ superconformal symmetry is a gauge symmetry. In order to explain how we can restore this symmetry we need to consider a real perturbation, *i.e.* we need to add to the (cc) perturbation its hermitian conjugate, which is then an (aa) perturbation. The above considerations imply that the gluing map for $G = G^+ + G^-$ changes by

$$\Delta G = 2\lambda\pi \left[G^-_{-1/2}\varphi_{h=1,q=2} + G^+_{-1/2}\varphi_{h=1,q=-2} \right] , \tag{3.13}$$

where $\varphi_{h=1,q=-2}$ is the field that appears in the even fusion rule of ϕ_a with $\bar{\phi}_a$. To restore the original gluing condition, we now turn on the boundary perturbation

$$\Delta S_{bd} = i\lambda \int dx \left[\varphi_{h=1,q=2} + \varphi_{h=1,q=-2} \right] . \tag{3.14}$$

The effect of boundary perturbations on the gluing map were analysed in [44]. The relevant term comes from the first order pole as the chiral field G approaches the boundary field in (3.14). Since $\varphi_{h=1,q=2}$ is a chiral primary field, while $\varphi_{h=1,q=-2}$ is anti-chiral, we have

$$G(z) \left[\varphi_{h=1,q=2} + \varphi_{h=1,q=-2} \right] = \frac{1}{z-x} \left[G^-_{-1/2}\varphi_{h=1,q=2} + G^+_{-1/2}\varphi_{h=1,q=-2} \right] + \text{regular} , \tag{3.15}$$

which therefore cancels precisely (3.13). This ties in precisely with what we found in the field theory approach where the boundary term (2.13) restored the $\mathcal{N} = 1$ supersymmetry.

To leading order this perturbation respects the conformal symmetry (since the boundary field is marginal). However, generically, the boundary field in (3.14) is not exactly marginal, and thus this perturbation will induce an RG flow to higher order. This should correspond to the RG flow of [41].

3.3 $\mathcal{N}=2$ B-type branes under (ac) deformations

It is expected [16] that the stability of B-type branes depends on the (ac) moduli. This has been made more precise in [18, 45, 19] where it was shown that (ac) perturbations can change the boundary condition for the spectral flow operator and hence possibly destroy the charge quantisation condition in the open string sector. As a consequence, a (ac) bulk perturbation may lead to the appearance of open string tachyons, triggering a decay of the brane at lines of marginal stability. Note that the perturbation by (ac) fields is rather different in nature than that by (cc) fields: the former influence the integer charge quantisation and lead to a potential breaking of conformal invariance, while world-sheet supersymmetry is always preserved. The latter can potentially break world-sheet supersymmetry, leading to obstructions, but leave the integer charge quantisation intact.

In the following we want to explain how these effects appear from our point of view. As before, the first step consists of analysing whether the bulk perturbation by $\widehat{\Phi}$ induces a marginal (or relevant) boundary field as it approaches the boundary. (If this happens, then the bulk perturbation will break the conformal invariance and hence induce an RG flow.) This question is now controlled by the fusion of a chiral primary field ϕ_c, with an anti-chiral primary field $\bar{\phi}_a$. Using the fusion rules of the $\mathcal{N}=2$ algebra [39, 40], one can show that only the even fusion rule can contribute,

$$\phi_c \otimes \bar{\phi}_a = [\varphi_{q=0}] \ , \tag{3.16}$$

but one cannot deduce any constraints about the conformal dimension of φ in general.[6] We therefore cannot exclude that such a flow is induced by the bulk perturbation. In fact, this flow is the world-sheet description of what happens on a line of marginal stability, and the line of marginal stability is the location in moduli space where the boundary field that is switched on by the bulk perturbation becomes marginal. As one crosses the line of marginal stability, the brane undergoes a non-trivial RG flow, and thus decays. At the end-point of the RG flow the would-be marginal boundary field is no longer marginal, and the new configuration is stable against further (ac) deformations.

Let us thus assume that we consider a configuration in the interior of the moduli space, away from any line of marginal stability. This means that (apart from the vacuum whose contribution we can regularise) the fusion rules (3.16) do not contain any relevant or marginal fields. It is then clear that (again up to possible vacuum contributions) the field $\widehat{\Phi}$ vanishes in the limit as it approaches the boundary.

We can then study the behaviour of the gluing conditions under the perturbation. The analysis is very similar to what was done before; in particular we now find that the potential contribution would have to come from $G^-_{-1/2}\varphi_{q=0}$ (for ΔG^+) and $G^+_{-1/2}\varphi_{q=0}$ (for ΔG^-). However, by our assumption above, neither of these fields survives in the $y \to 0$ limit. Thus we conclude that none of the gluing conditions are modified, so that no obstruction occurs. This explains, from a world-sheet point of view, why (ac) perturbations of B-type branes do not break supersymmetry, in agreement with the above expectations.

[6] We shall see in the examples below that non-trivial relevant fields do indeed appear.

3.4 The $\mathcal{N} = 4$ analysis

After this $\mathcal{N} = 2$ discussion we now want to understand to which extent these statements are modified for $\mathcal{N} = 4$ superconformal branes. Before we discuss the details, there is one general point we should stress. Suppose we are given some boundary condition that preserves the $\mathcal{N} = 4$ superconformal algebra up to some automorphism. We may absorb this automorphism into a redefinition of the left- or right-moving $\mathcal{N} = 4$ generators, and hence assume that the boundary condition preserves the $\mathcal{N} = 4$ algebra without any automorphism. With respect to the usual $\mathcal{N} = 2$ subalgebra, the resulting D-brane is then a B-type D-brane.

We are interested in studying the behaviour of this D-brane under bulk deformations (that preserve the $\mathcal{N} = (4,4)$ superconformal symmetry in the bulk). As we have mentioned before, such bulk deformations are of four different types: (cc), (ac,), (ca) or (aa). In an $\mathcal{N} = (2,2)$ theory these four fields lie in *different* $\mathcal{N} = (2,2)$ multiplets; however in the $\mathcal{N} = (4,4)$ case they always combine into one common $\mathcal{N} = (4,4)$ multiplet.

One may now be tempted to believe that the $\mathcal{N} = (4,4)$ symmetry would guarantee that the above brane behaves the same way under a (cc) perturbation, as under an (ac) perturbation, say. However, this is not true. Once we consider a brane that preserves the $\mathcal{N} = 2$ subalgebra without any automorphism (and hence is a B-type brane with respect to the $\mathcal{N} = 2$ subalgebra) then the $\mathcal{N} = 4$ automorphisms that preserve this property are the 'outer automorphisms' of the $\mathcal{N} = 4$ algebra that rotate the $G^{\pm}$ and $G'^{\pm}$ modes into one another. However, these automorphisms do *not* map a (cc) field to an (ac) field (since they do not modify the separate U(1)-charges). Put differently, the $\mathcal{N} = (4,4)$ automorphism that relates a (cc) perturbation to an (ac) perturbation also maps a B-type brane (with respect to the $\mathcal{N} = 2$ subalgebra) to an A-type brane (with respect to the same $\mathcal{N} = 2$ algebra). Thus, even in the situation with $\mathcal{N} = (4,4)$ supersymmetry, there are two cases to consider. We shall now discuss them in turn.

3.4.1 (cc) perturbation of B-type branes

To start with we just employ the $\mathcal{N} = (2,2)$ superconformal symmetry. Thus we can repeat the analysis of section 3.2. In particular, it therefore follows that the (cc) perturbation will not break the conformal symmetry. On the other hand, as we saw in section 3.2.1, generically the $\mathcal{N} = 2$ superconformal symmetry may be broken. In fact, the term that modifies the gluing condition for the G^+ generator is precisely given by (3.11). Note that this term only breaks the symmetry if the right-hand-side of (3.11) is *not* part of the symmetry algebra of the theory, *i.e.* if $\varphi_{h=1,q=2}$ is not part of the symmetry algebra. Otherwise, (3.11) only means that the gluing condition for G^+ is suitably modified.

If we are just considering an $\mathcal{N} = 2$ superconformal field theory, then the chiral algebra does not contain a field of $h = 1$ and $q = 2$, and hence the symmetry is broken (provided that $\varphi_{h=1,q=2} \neq 0$). In fact, *if* the $\mathcal{N} = 2$ superconformal field theory contains such a chiral field (as well as its hermitian conjugate), then the symmetry is enlarged, and one finds that the resulting chiral algebra must contain an algebra with $\mathcal{N} = 4$ superconformal symmetry (where the additional supercharges are obtained from $G^{\mp}$ by the action of $\varphi_{h=1,q=\pm 2} \cong J^{\pm}$). Conversely, if the boundary theory preserves the $\mathcal{N} = 4$ superconformal symmetry for $c = 6$, then the representation theory of the $\mathcal{N} = 4$ algebra

at $c = 6$ ($k = 1$) implies that $\varphi_{h=1,q=2} \cong J^+$ [46]. Thus in this case, the gluing condition (3.11) is modified by a field in the chiral algebra itself.[7] Then the symmetry is not broken but only the gluing condition is modified. Since the same argument also holds for G'^+ (and similarly for G^- and G'^-), it then follows that the full $\mathcal{N} = 4$ superconformal symmetry is preserved in this case. This shows that $\mathcal{N} = 4$ B-type branes are never obstructed against (cc) perturbations. This is in nice agreement with what we found in the superspace analysis of section 2.

Although the $\mathcal{N} = 4$ superconformal symmetry is preserved, the gluing condition for the $\mathcal{N} = 1$ subalgebra is generically broken. This can be adjusted as in section 3.2.2. In the $\mathcal{N} = 4$ case, the boundary field in (3.14) is exactly marginal, and hence no higher order RG flow is induced.

3.4.2 (ac) perturbations of B-type branes

The situation where we perturb an $\mathcal{N} = 4$ B-type brane by an (ac) perturbation is more subtle. From the point of view of the $\mathcal{N} = 2$ analysis, we saw that the bulk perturbation may switch on a marginal field on the boundary. The geometric interpretation of this phenomenon was that this happens on a line of marginal stability. One may be tempted to believe that the $\mathcal{N} = 4$ representation theory should imply that such a phenomenon cannot happen. However, this does not seem to be the case (see appendix B.2). In fact, there is an explicit example (see section 4.2) where a non-trivial boundary field on an $\mathcal{N} = 4$ preserving D-brane is switched on by an (ac) bulk deformation. At least in that example, this boundary field will induce an RG flow on the boundary (and thus modify the D-brane significantly) but it does not seem to lead to a 'brane decay' and therefore should probably not be interpreted as indicating a 'line of marginal stability'. However, from the point of view of conformal perturbation theory, the fact that no 'brane decay' takes place cannot be seen at leading order in perturbation theory, and it is therefore difficult — even in the situation with $\mathcal{N} = 4$ — to make definite general predictions about the absence or otherwise of possible brane decays for $\mathcal{N} = 4$ branes.

Finally, if we sit at a generic point in the (ac) moduli space, *i.e.* not on a line of marginal stability, then the same conclusion as in section 3.3 applies, and we conclude that there is no obstruction.

4 Some examples

In this section we want to present two simple examples that illustrate some of the claims we have been making above. In particular, we want to exemplify the mechanisms behind the obstruction of B-type $\mathcal{N} = 2$ branes under (cc) deformations in a simple torus example (section 4.1). In section 4.2 we shall then discuss $\mathcal{N} = 4$ branes on K3 at the Gepner point and show that under (ac) perturbations non-trivial boundary fields may be switched on.

[7] For $\varphi_{h=1,q=2} = J^+$, the field $G^-_{-1/2}J^+$ is another supercurrent (namely $G^-_{-1/2}J^+ = -G'^+$), as follows from the $\mathcal{N} = 4$ commutation relations (B.1).

4.1 A simple torus example

To illustrate some aspects of our $\mathcal{N} = 2$ analysis, let us consider the free theory on $T^4 = T^2 \times T^2$, where initially both tori are taken to be square tori of the same size. On each torus there is a complex boson and a complex fermion whose (left-moving) modes we denote by $\alpha_n^{(i)}$, $\bar{\alpha}_n^{(i)}$ and $\psi_m^{(i)}$, $\bar{\psi}_m^{(i)}$, where $i = 1, 2$ labels the two tori. The corresponding right-moving modes will be denoted by a tilde. All of these modes satisfy the usual (anti)-commutation relations. The theory has actually $\mathcal{N} = 4$ supersymmetry — see *e.g.* [43] for the construction of the corresponding superconformal algebra. Although this means that there will be no actual obstructions, all the effects described above occur.

We are interested in the gluing condition determined by

$$\begin{array}{rclcrcl} \tilde{\alpha}^{(1)} & = & \omega_\lambda(\alpha^{(1)}) \ , & \qquad & \tilde{\bar{\alpha}}^{(1)} & = & \omega_\lambda(\bar{\alpha}^{(1)}) \\ \tilde{\alpha}^{(2)} & = & \omega_\lambda(\alpha^{(2)}) \ , & \qquad & \tilde{\bar{\alpha}}^{(2)} & = & \omega_\lambda(\bar{\alpha}^{(2)}) \ , \end{array} \tag{4.1}$$

where ω_λ is defined as

$$\begin{array}{ll} \omega_\lambda(\alpha^{(1)}) = \cos 2\theta_\lambda \bar{\alpha}^{(1)} + \sin 2\theta_\lambda \alpha^{(2)} & \omega_\lambda(\bar{\alpha}^{(1)}) = \cos 2\theta_\lambda \alpha^{(1)} + \sin 2\theta_\lambda \bar{\alpha}^{(2)} \\ \omega_\lambda(\alpha^{(2)}) = \cos 2\theta_\lambda \bar{\alpha}^{(2)} - \sin 2\theta_\lambda \alpha^{(1)} & \omega_\lambda(\bar{\alpha}^{(2)}) = \cos 2\theta_\lambda \alpha^{(2)} - \sin 2\theta_\lambda \bar{\alpha}^{(1)} \ . \end{array} \tag{4.2}$$

We also impose similar gluing conditions for the fermions. For the initial square tori $\theta_\lambda = \frac{\pi}{4}$ is an allowed boundary condition, and one easily shows that it satisfies B-type gluing conditions for the usual (diagonal) $\mathcal{N} = 2$ algebra (see *e.g.* [43]).

Let us now perturb the theory by the (cc) and (aa) fields

$$\phi_{cc} = \bar{\psi}^{(1)}_{-1/2} \tilde{\bar{\psi}}^{(1)}_{-1/2} |0\rangle \ , \qquad \phi_{aa} = \psi^{(1)}_{-1/2} \tilde{\psi}^{(1)}_{-1/2} |0\rangle \ , \tag{4.3}$$

so that the full perturbation fields are

$$\begin{array}{rcl} \Phi_{cc} & = & G^-_{-1/2} \tilde{G}^-_{-1/2} \phi_{c,c} = -2\bar{\alpha}^{(1)}_{-1} \tilde{\bar{\alpha}}^{(1)}_{-1} |0\rangle \ , \\ \Phi_{aa} & = & G^+_{-1/2} \bar{G}^+_{-1/2} \phi_{a,a} = -2\alpha^{(1)}_{-1} \tilde{\alpha}^{(1)}_{-1} |0\rangle \ . \end{array} \tag{4.4}$$

The corresponding hermitian fields are then described by the two linear combinations

$$\delta S_1 = \int d^2 z \, (\Phi_{cc} + \Phi_{aa}) \ , \qquad \text{and} \qquad \delta S_2 = i \int d^2 z \, (\Phi_{cc} - \Phi_{aa}) \ . \tag{4.5}$$

Physically, in terms of the moduli of the torus, δS_1 corresponds to switching on the combination $(g_{11} - g_{22})$ of the diagonal elements of the first torus metric, while δS_2 corresponds to modifying g_{12} of the first torus.[8]

Given the explicit form of the perturbing fields (4.4) as well as the boundary conditions (4.2), it is easy to see that Φ_{cc} only switches on a boundary field with $h = 2$ as it approaches the boundary, and likewise for Φ_{aa}. Thus the perturbation does not break the conformal invariance, in agreement with the analysis of section 3.2. The effect on the gluing map is very similar to what has been analysed in [36]. To first order it equals

$$\delta\omega_\lambda(\alpha^{(1)}) = 2 \cos 2\theta_\lambda \alpha^{(2)} \ , \tag{4.6}$$

[8]The remaining moduli of the first torus (the volume form and the anti-symmetric B-field) correspond to deformations involving the (ca) and (ac) fields. In addition there are also similar moduli corresponding to changing the second torus.

and actually this can be integrated up for arbitrary finite λ. This leads to the differential equation

$$\frac{d}{d\lambda}\omega_\lambda(\alpha^{(1)}) = \pi \sin^2 2\theta_\lambda \bar{\alpha}^{(1)} - \pi \cos 2\theta_\lambda \sin 2\theta_\lambda \alpha^{(2)} \ , \tag{4.7}$$

which is solved by $\omega_\lambda(\alpha^{(1)})$ if θ_λ satisfies

$$\dot{\theta}_\lambda = -\frac{\pi}{2} \sin 2\theta_\lambda \ . \tag{4.8}$$

Similar differential equations can be obtained for the other bosonic modes as well, and all of them just correspond to modifying ω_λ as in (4.8).

It is also straightforward to see that the gluing condition for the fermions is not affected by the perturbation. Since the $\mathcal{N}=2$ supercharges are bilinear combinations of bosons and fermions, it is then straightforward to determine their gluing conditions. One finds that for $\lambda \neq 0$ they do not close any longer on the $\mathcal{N}=2$ subalgebra, but rather involve also the other supercharges of the $\mathcal{N}=4$ algebra [43]. Among other things this then also implies that the $\mathcal{N}=1$ gluing conditions are modified. To restore them we proceed as in section 3.2.2. and switch on the boundary perturbation (3.14), which in the current context takes the form

$$i\lambda \int_{\mathbb{R}} dx \ (\phi_{c,c}(x) - \phi_{a,a}(x)) = i\lambda \int_{\mathbb{R}} dx \ \left(\psi^{(1)}\omega_\lambda(\psi^{(1)}) + \bar{\psi}^{(1)}\omega_\lambda(\bar{\psi}^{(1)})\right) \ . \tag{4.9}$$

This field induces a change in the gluing condition for the fermions (while it does not affect the bosons); following [44], it leads to

$$\delta\omega_\lambda(\psi) = -2\pi\delta\lambda \left[\left(\psi^{(1)}\omega_\lambda(\psi^{(1)}) + \bar{\psi}^{(1)}\omega_\lambda(\bar{\psi}^{(1)})\right)_0, \omega_\lambda(\psi)\right] \ . \tag{4.10}$$

In particular, this therefore guarantees that the fermions continue to satisfy the same gluing condition as the bosons, *i.e.* the one determined in terms of ω_λ. It is then in particular manifest that the original $\mathcal{N}=1$ is preserved. While this restores the $\mathcal{N}=1$ gluing conditions, it does not do so for the $\mathcal{N}=2$ B-type gluing conditions. However, the resulting boundary conditions still preserves the $\mathcal{N}=4$ symmetry [43], in agreement with the analysis of section 3.4.1.

4.2 K3 at the Fermat point

Another interesting example is the non-linear sigma model on K3, whose infrared theory defines an $\mathcal{N}=(4,4)$ superconformal theory. More precisely, we will concentrate on the Fermat point given by the zero locus of $W = x_1^4 + x_2^4 + x_3^4 + x_4^4$, where there are (at least) two alternative formulations: as a Gepner model [47, 48], associated to four copies of the $k=2$ $\mathcal{N}=2$ minimal model, or as the torus orbifold $T^4/\mathbb{Z}_4$. In the Gepner construction only the $\mathcal{N}=(2,2)$ superconformal symmetry is manifest; the additional $SU(2)$ currents that enhance it to $\mathcal{N}=(4,4)$ are explicitly given as

$$J^+ = \bigotimes_{i=1}^r (0,2,2) \otimes \overline{(0,0,0)} = \bigotimes_{i=1}^r (k_i, -k_i, 0) \otimes \overline{(0,0,0)} \ , \tag{4.1}$$

$$J^- = \bigotimes_{i=1}^r (0,-2,2) \otimes \overline{(0,0,0)} = \bigotimes_{i=1}^r (k_i, k_i, 0) \otimes \overline{(0,0,0)} \ , \tag{4.2}$$

and similarly for the right-movers. Here, as in the following, we shall use the conventions of [43].

We shall consider the B-type D-brane that is formulated as a permutation brane [49] associated to the (12)(34) permutation. In the orbifold description, this brane corresponds to a superposition of two D2-branes, one with Neumann directions along x^1 and x^3, and the other with Neumann directions along x^2 and x^4. This configuration is then invariant under the $\mathbb{Z}_4$ rotation action. (For details about these branes and their identification see [43].) Since this boundary condition actually preserves the $\mathcal{N} = 4$ superconformal symmetry, we should be able to test our predictions of section 3.4.

First of all, we have checked that the $h = 1, q = 2$ field that is potentially switched on by the (cc) perturbations is always the J^+ current on the boundary, *i.e.* the restriction of the bulk J^+ current (4.1) to the boundary. This shows that this brane is not obstructed against any of the (cc) perturbations.

Incidentally, a similar analysis for an $\mathcal{N} = 2$ brane in a $c = 9$ Calabi-Yau Gepner model typically leads to a boundary field that is not exactly marginal. For example, consider the (12)(35)(4) brane in the $(k = 3)^{\otimes 5}$ model, and perturb by the (cc) field

$$\begin{aligned} \phi \;=\; & \Big((3,-3,0) \otimes \overline{(3,-3,0)}\Big) \otimes \Big((0,0,0) \otimes \overline{(0,0,0)}\Big) \otimes \Big((2,-2,0) \otimes \overline{(2,-2,0)}\Big) \\ & \otimes\Big((0,0,0) \otimes \overline{(0,0,0)}\Big) \otimes \Big((0,0,0) \otimes \overline{(0,0,0)}\Big) \; . \end{aligned} \tag{4.3}$$

On the permutation brane this bulk field induces the boundary field

$$\Psi = (3,-3,0) \otimes (3,-3,0) \otimes (2,-2,0) \otimes (0,0,0) \otimes (2,-2,0) \; , \tag{4.4}$$

which of course has $h = 1$ and $q = 2$. The OPE of Ψ with itself then contains

$$\Psi\Psi \sim (3,-1,2) \otimes (3,-1,2) \otimes (3,1,2) \otimes (0,0,0) \otimes (3,1,2) \; , \tag{4.5}$$

which has $h = \frac{14}{5}$, thus showing that Ψ does not belong to the chiral algebra, *i.e.* that it is not exactly marginal. We would therefore expect that a non-trivial boundary flow is induced, and this is indeed what was found in [41].

Let us now return to the case of the (12)(34) permutation brane on the $(k = 2)^{\otimes 4}$ model of K3, and analyse the behaviour under an (ac) perturbations. As an example of such a perturbation we consider the bulk field

$$\phi = \Big((0,0,0)\otimes\overline{(0,2,2)}\Big)\otimes\Big((1,1,0)\otimes\overline{(1,3,2)}\Big)\otimes\Big((1,1,0)\otimes\overline{(1,3,2)}\Big)\otimes\Big((2,2,0)\otimes\overline{(2,4,2)}\Big) \tag{4.6}$$

from the first twisted sector of the Gepner orbifold. (From the point of view of the torus orbifold, this field arises in the first twisted sector of the $\mathbb{Z}_4$ orbifold.) Again, using the permutation gluing condition, it is easy to see that ϕ induces the boundary field

$$\Psi = (1,-1,0) \otimes (1,-1,0) \otimes (1,1,0) \otimes (1,1,0) \tag{4.7}$$

on the boundary. This field has $q = 0$ and $h = 1/2$, and thus is an example of a non-trivial (relevant) field appearing in the fusion rule (B.16). From the orbifold point of view, this field has also a simple interpretation: it is precisely the lowest mode of the

open string between the two D2-branes. (Note that from the point of view of the orbifold theory this is not a boundary changing field since only the superpositions of D2-branes is orbifold invariant. In fact, since the bulk field comes from the first twisted sector, it transforms with a primitive fourth root of unity under the orbifold quantum symmetry. The corresponding boundary field must then transform with the complex conjugate phase under this quantum symmetry since otherwise the bulk-boundary correlator would vanish. But this means that it must be a boundary changing field since these get projected out when one performs the quantum symmetry orbifold to return to the original theory.)

Geometrically, this boundary field will probably modify the way the two D2-branes meet at the orbifold singularity — the bulk field in question describes after all one of the blowing-up modes of the orbifold fixed point. However, we do not think this will lead to any form of 'brane decay' since there are probably no configurations of two D-branes that have the same RR charge and tension. Thus we should probably not interpret this as an example of a line of marginal stability. However, from the perturbed conformal field theory point of view, this phenomenon looks identical to what happens, say, in the $\mathcal{N}=2$ context where there are lines of marginal stability.

5 Conclusions

In this paper we have studied how the obstructions of supersymmetric branes may be understood from a world-sheet perspective. In particular, we have explained how the (possible) obstruction of an $\mathcal{N}=2$ B-type brane under a (cc) perturbation arises in the superspace field theory language, and from the point of view of perturbed conformal field theory. We have also explained that this phenomenon does not occur for $\mathcal{N}=4$ superconformal branes at $c=6$, in particular for half-supersymmetric D-branes on K3.

While our analysis for the perturbations of B-type branes under (cc) deformations is fairly complete, the behaviour of B-type branes under (ac) deformations (or A-type branes under (cc) deformations) is more subtle. In the $\mathcal{N}=2$ context it is believed that there are no actual obstructions, but that branes may decay along lines of marginal stability. This phenomenon is not visible in the superspace field theory language. However, we can understand the origin of it in the conformal field theory analysis where it corresponds to the situation when an irrelevant boundary field (that is switched on by the bulk deformation) becomes marginal (see section 3.3). In the case of half-BPS branes in $\mathcal{N}=4$ theories, there is evidence that lines of marginal stability do not exist. This expectation is mainly based on the fact that the index that counts half-BPS states is constant on the full moduli space. However, the analysis of section 4.2 seems to suggest that at least relevant boundary fields may be switched on by bulk deformations, even in the $\mathcal{N}=4$ case. Thus on the basis of our leading order conformal field theory analysis it does not appear to be possible to detect any difference between the $\mathcal{N}=2$ and $\mathcal{N}=4$ case in this respect. It would be interesting to understand whether the higher order behaviour is different between the $\mathcal{N}=2$ and the $\mathcal{N}=4$ case. It would also be very interesting to relate this world-sheet analysis with recent progress for 'wall-crossing formulae' of $\mathcal{N}=2$ branes, see for example [50].

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A Superspace conventions

A.1 $\mathcal{N}=2$ superspace conventions

The $\mathcal{N}=2$ supersymmetry transformations are generated by the following supercharges

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar\theta^\pm\partial_\pm \ , \qquad \bar{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\bar\theta^\pm} - i\theta^\pm\partial_\pm \ .$$

Here $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$ and $x^\pm = x^0 \pm x^1$ are the usual light-cone coordinates. One easily checks that the only non-trivial anti-commutators of these generators are

$$\{\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm\} = -2i\partial_\pm \ . \tag{A.1}$$

For the following it is also useful to introduce the corresponding spinor derivatives

$$\mathcal{D}_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar\theta^\pm\partial_\pm \ , \qquad \bar{\mathcal{D}}_\pm = -\frac{\partial}{\partial\bar\theta^\pm} + i\theta^\pm\partial_\pm \ .$$

One easily confirms that all of these generators anti-commute with all the $\mathcal{Q}_\pm$ and $\bar{\mathcal{Q}}_\pm$. Furthermore, the only non-trivial anti-commutator involving the $\mathcal{D}$ generators is

$$\{\mathcal{D}_\pm, \bar{\mathcal{D}}_\pm\} = +2i\partial_\pm \ . \tag{A.2}$$

With the help of these operators we can now define what we mean by a chiral superfield Φ: this is a superfield that satisfies

$$\bar{\mathcal{D}}_\pm\Phi = 0 \ . \tag{A.3}$$

Every chiral superfield can be written as

$$\Phi(x^\pm,\theta^\pm,\bar\theta^\pm) = \phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_-(y^\pm) + \theta^+\theta^-F(y^\pm) \ , \tag{A.4}$$

where

$$y^\pm = x^\pm - i\theta^\pm\bar\theta^\pm \ . \tag{A.5}$$

Indeed, one easily checks that Φ in (A.4) satisfies (A.3). The complex conjugate of a chiral superfield Φ is an anti-chiral superfield $\bar\Phi$; anti-chiral superfields are characterised by $\mathcal{D}_\pm\bar\Phi = 0$, and they have an expansion

$$\bar\Phi(x^\pm,\theta^\pm,\bar\theta^\pm) = \bar\phi(\bar y^\pm) + \bar\theta^+\bar\psi_+(\bar y^\pm) + \bar\theta^-\bar\psi_-(\bar y^\pm) + \bar\theta^+\bar\theta^-\bar F(\bar y^\pm) \ , \tag{A.6}$$

where

$$\bar{y}^{\pm} = x^{\pm} + i\theta^{\pm}\bar{\theta}^{\pm} \ . \tag{A.7}$$

A twisted chiral superfield U is a superfield that satisfies

$$\bar{\mathcal{D}}_+ U = \mathcal{D}_- U = 0 \ . \tag{A.8}$$

Every twisted chiral superfield can be written as

$$U(x^{\pm},\theta^{\pm},\bar{\theta}^{\pm}) = v(\tilde{y}^{\pm}) + \theta^+\chi_+(\tilde{y}^{\pm}) + \bar{\theta}^-\bar{\chi}_-(\tilde{y}^{\pm}) + \theta^+\bar{\theta}^- E(\tilde{y}^{\pm}) \ , \tag{A.9}$$

where

$$\tilde{y}^{\pm} = x^{\pm} \mp i\theta^{\pm}\bar{\theta}^{\pm} \ . \tag{A.10}$$

Finally, a general twisted anti-chiral superfield that is characterised by

$$\mathcal{D}_+ \bar{U} = \bar{\mathcal{D}}_- \bar{U} = 0 \tag{A.11}$$

can always be written as

$$\bar{U}(x^{\pm},\theta^{\pm},\bar{\theta}^{\pm}) = \bar{v}(\hat{y}^{\pm}) + \bar{\theta}^+\bar{\chi}_+(\hat{y}^{\pm}) + \theta^-\chi_-(\hat{y}^{\pm}) + \bar{\theta}^+\theta^-\bar{E}(\hat{y}^{\pm}) \ , \tag{A.12}$$

where

$$\hat{y}^{\pm} = x^{\pm} \pm i\theta^{\pm}\bar{\theta}^{\pm} \ . \tag{A.13}$$

A.2 $\mathcal{N}=4$ superspace conventions

In what follows it will be understood that shifting of $SU(2)$ indices is accomplished with the help of ϵ-symbols. In the $\mathcal{N}=4$ standard superspace (2.19), supersymmetry transformations will be generated by

$$\mathcal{Q}^i = \frac{\partial}{\partial\theta_i} + i\bar{\theta}^i\partial_+ \ , \qquad \text{and} \qquad \bar{\mathcal{Q}}_i = -\frac{\partial}{\partial\bar{\theta}^i} - i\theta_i\partial_+ \ , \tag{A.14}$$

$$\mathcal{Q}^{\bar{\imath}} = \frac{\partial}{\partial\theta_{\bar{\imath}}} + i\bar{\theta}^{\bar{\imath}}\partial_- \ , \qquad \text{and} \qquad \bar{\mathcal{Q}}_{\bar{\imath}} = -\frac{\partial}{\partial\bar{\theta}^{\bar{\imath}}} - i\theta_{\bar{\imath}}\partial_- \ , \tag{A.15}$$

with the following non-trivial anti-commutation relations

$$\{\mathcal{Q}^i,\bar{\mathcal{Q}}_j\} = -2i\delta^i_j\partial_+ \ , \qquad \{\mathcal{Q}^{\bar{\imath}},\bar{\mathcal{Q}}_{\bar{\jmath}}\} = -2i\delta^{\bar{\imath}}_{\bar{\jmath}}\partial_- \ . \tag{A.16}$$

Similarly, we can also introduce the corresponding spinor derivatives which take the form

$$\mathcal{D}^i = \frac{\partial}{\partial\theta_i} - i\bar{\theta}^i\partial_+ \ , \qquad \text{and} \qquad \bar{\mathcal{D}}_i = -\frac{\partial}{\partial\bar{\theta}^i} + i\theta_i\partial_+ \ , \tag{A.17}$$

$$\mathcal{D}^{\bar{\imath}} = \frac{\partial}{\partial\theta_{\bar{\imath}}} - i\bar{\theta}^{\bar{\imath}}\partial_- \ , \qquad \text{and} \qquad \bar{\mathcal{D}}_{\bar{\imath}} = -\frac{\partial}{\partial\bar{\theta}^{\bar{\imath}}} + i\theta_{\bar{\imath}}\partial_- \ . \tag{A.18}$$

The only non-vanishing anti-commutation relations involving $\mathcal{D}$s are given by

$$\{\mathcal{D}^i,\bar{\mathcal{D}}_j\} = 2i\delta^i_j\partial_+ \ , \qquad \{\mathcal{D}^{\bar{\imath}},\bar{\mathcal{D}}_{\bar{\jmath}}\} = 2i\delta^{\bar{\imath}}_{\bar{\jmath}}\partial_- \ . \tag{A.19}$$

In order to avoid cluttering our formulae we combine all spinors into doublets with respect to $SU(2)_c$. First of all, we write the Grassmann variables as[9]

$$\theta_i^a = \begin{pmatrix} \theta_i \\ \bar{\theta}_i \end{pmatrix}, \qquad \text{and} \qquad \theta_{\bar{\imath}}^{\bar{a}} = \begin{pmatrix} \theta_{\bar{\imath}} \\ \bar{\theta}_{\bar{\imath}} \end{pmatrix}, \tag{A.20}$$

as well as the supercharges

$$\mathcal{Q}_a^i = \begin{pmatrix} \mathcal{Q}^i \\ \bar{\mathcal{Q}}^i \end{pmatrix}, \qquad \mathcal{Q}_{\bar{a}}^{\bar{\imath}} = \begin{pmatrix} \mathcal{Q}^{\bar{\imath}} \\ \bar{\mathcal{Q}}^{\bar{\imath}} \end{pmatrix}, \tag{A.21}$$

and finally also the spinor derivatives

$$\mathcal{D}_a^i = \begin{pmatrix} \mathcal{D}^i \\ \bar{\mathcal{D}}^i \end{pmatrix}, \qquad \mathcal{D}_{\bar{a}}^{\bar{\imath}} = \begin{pmatrix} \mathcal{D}^{\bar{\imath}} \\ \bar{\mathcal{D}}^{\bar{\imath}} \end{pmatrix}. \tag{A.22}$$

An example of a superfield living on the superspace (2.19) is the $\mathcal{N} = (4,4)$ twisted multiplet (see [20, 24, 25]) whose Grassmann expansion reads

$$\Phi^{i\bar{\imath}} = \varphi^{i\bar{\imath}} + 2\theta^{a,i}\psi_a^{\bar{\imath}} + 2\theta^{\bar{a},\bar{\imath}}\psi_{\bar{a}}^i + 2\theta^{a,i}\theta^{\bar{a},\bar{\imath}}F_{a\bar{a}} + \text{derivatives}\ . \tag{A.23}$$

The lowest component of this field is given by a quartet of (pseudo-real) scalar fields, which satisfy the relation

$$\overline{(\varphi^{i\bar{\imath}})} = \epsilon_{ij}\epsilon_{\bar{\imath}\bar{\jmath}}\varphi^{j\bar{\jmath}}\ . \tag{A.24}$$

A.3 Harmonic superspace conventions

A.3.1 Supercharges and spinor derivatives

In (2.21) we have introduced harmonic coordinates

$$\{u_i^{(\pm,0)}\} \in \frac{SU(2)_L}{U(1)_L}, \qquad \text{and} \qquad \{u_{\bar{\imath}}^{(0,\pm)}\} \in \frac{SU(2)_R}{U(1)_R}, \tag{A.25}$$

where $SU(2)_L$ and $SU(2)_R$ will be identified with the left- and right-moving $SU(2)_f$, respectively. Explicitly we can write the harmonic variables in the following matrix form

$$\begin{pmatrix} u_1^{(-,0)} & u_1^{(+,0)} \\ u_2^{(-,0)} & u_2^{(+,0)} \end{pmatrix} \in SU(2)_L\ , \qquad \text{and} \qquad \begin{pmatrix} u_1^{(0,-)} & u_1^{(0,+)} \\ u_2^{(0,-)} & u_2^{(0,+)} \end{pmatrix} \in SU(2)_R\ . \tag{A.26}$$

Acting with $SU(2)_L$ on the left, we see that both $u_i^{(+,0)}$ and $u_i^{(-,0)}$ are doublets. The $\pm$ indicates the $U(1)_L$ charge, as can be seen by acting with $U(1)_L \subset SU(2)_L$ on the right. Similar statements obviously hold for the right moving expressions. We define raising and lowering of indices via complex conjugation,

$$\begin{aligned} \overline{u^{(+,0),i}} &= u_i^{(-,0)} \\ \overline{u_i^{(+,0)}} &= -u^{(-,0),i} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overline{u^{(0,+),\bar{\imath}}} &= u_{\bar{\imath}}^{(0,-)} \\ \overline{u_{\bar{\imath}}^{(0,+)}} &= -u^{(0,-),\bar{\imath}}\ . \end{aligned} \tag{A.27}$$

[9] Notice that they transform as $(\mathbf{2},\mathbf{2})$ under $SU(2)_c \times SU(2)_f$.

It then follows that

$$u^{(+,0),i}u_i^{(-,0)} = 1 \quad \text{and} \quad u^{(0,+),\bar{\imath}}u_{\bar{\imath}}^{(0,-)} = 1 \ , \quad \text{(unit determinant condition)} \ , \tag{A.28}$$

and, using the convention $\epsilon_{12} = -1$,

$$\epsilon_{ij} = u_i^{(+,0)}u_j^{(-,0)} - u_i^{(-,0)}u_j^{(+,0)} \ , \quad \text{and} \quad \epsilon_{\bar{\imath}\bar{\jmath}} = u_{\bar{\imath}}^{(0,+)}u_{\bar{\jmath}}^{(0,-)} - u_{\bar{\imath}}^{(0,-)}u_{\bar{\jmath}}^{(0,+)} \ . \tag{A.29}$$

These relations entail that not all of the harmonic variables are independent of each other. This has in particular the effect that naïve partial derivatives with respect to u are not allowed. Instead one has to use covariant harmonic derivatives, which take the following form (see for example [24, 30])

$$D^{(\pm 2,0)} = u_i^{(\pm,0)}\frac{\partial}{\partial u_i^{(\mp,0)}} \ , \qquad D^{(0,\pm 2)} = u_i^{(0,\pm)}\frac{\partial}{\partial u_i^{(0,\mp)}} \ , \tag{A.30}$$

$$D_{(L)}^{(0,0)} = u_i^{(\pm,0)}\frac{\partial}{\partial u_i^{(\pm,0)}} \ , \qquad D_{(R)}^{(0,0)} = u_i^{(0,\pm)}\frac{\partial}{\partial u_i^{(0,\pm)}} \ . \tag{A.31}$$

One can easily convince oneself that these derivatives respect the relation (A.28).

The integration over the harmonic variables is simply an integration on the sphere which picks the singlet piece of the harmonic expansion of the integrand. In particular we have

$$\int du^{(\pm,0)}1 = \int du^{(0,\pm)}1 = 1 \ , \tag{A.32}$$

while we get on the other hand (for $m \neq 0 \neq n$)

$$\int du^{(\pm,0)}u_{(i_1}^{(+,0)}\dots u_{i_n}^{(+,0)}u_{j_1}^{(-,0)}\dots u_{j_m)}^{(-,0)} = 0 \ , \tag{A.33}$$

$$\int du^{(0,\pm)}u_{(\bar{\imath}_1}^{(0,+)}\dots u_{\bar{\imath}_n}^{(0,+)}u_{\bar{\jmath}_1}^{(0,-)}\dots u_{\bar{\jmath}_m)}^{(0,-)} = 0 \ . \tag{A.34}$$

The supersymmetries in the formulation of (2.20) are generated by the supercharges of standard $\mathcal{N} = 4$ superspace (*i.e.* (A.15)) projected by the harmonic coordinates

$$\begin{aligned} \mathcal{Q}_a^{(\pm,0)} &= u_i^{(\pm,0)}\mathcal{Q}_a^i = \mp\frac{\partial}{\partial\theta^{a(\mp,0)}} + i\bar{\theta}_a^{(\pm,0)}\partial_+ \ , \\ \mathcal{Q}_{\bar{a}}^{(0,\pm)} &= u_{\bar{\imath}}^{(0,\pm)}\mathcal{Q}_{\bar{a}}^{\bar{\imath}} = \mp\frac{\partial}{\partial\theta^{\bar{a}(0,\mp)}} + i\bar{\theta}_{\bar{a}}^{(0,\pm)}\partial_- \ . \end{aligned} \tag{A.35}$$

Using (A.16), the projected commutation relations become

$$\{\mathcal{Q}_a^{(\pm,0)}, \mathcal{Q}_b^{(\mp,0)}\} = \mp 2i\,\epsilon_{ab}\,\partial_+ \ , \qquad \{\mathcal{Q}_{\bar{a}}^{(0,\pm)}, \mathcal{Q}_{\bar{b}}^{(0,\mp)}\} = \mp 2i\,\epsilon_{\bar{a}\bar{b}}\,\partial_- \ . \tag{A.36}$$

By projecting (A.18) we can in the same manner also define the projected spinor derivatives

$$\begin{aligned} \mathcal{D}_a^{(\pm,0)} &= u_i^{(\pm,0)}\mathcal{D}_a^i = \mp\frac{\partial}{\partial\theta^{a(\mp,0)}} - i\bar{\theta}_a^{(\pm,0)}\partial_+ \ , \\ \mathcal{D}_{\bar{a}}^{(0,\pm)} &= u_{\bar{\imath}}^{(0,\pm)}\mathcal{D}_{\bar{a}}^{\bar{\imath}} = \mp\frac{\partial}{\partial\theta^{\bar{a}(0,\mp)}} - i\bar{\theta}_{\bar{a}}^{(0,\pm)}\partial_- \ , \end{aligned} \tag{A.37}$$

which (in view of (A.19)) satisfy

$$\{\mathcal{D}_a^{(\pm,0)},\mathcal{D}_b^{(\mp,0)}\} = \pm\, 2i\,\epsilon_{ab}\,\partial_+ \ , \qquad\qquad \{\mathcal{D}_{\bar a}^{(0,\pm)},\mathcal{D}_{\bar b}^{(0,\mp)}\} = \pm\, 2i\,\epsilon_{\bar a\bar b}\,\partial_- \ . \tag{A.38}$$

For further convenience let us also introduce the so called analytic basis, in which we redefine the space-time coordinates in the following manner

$$z^+ = x^+ - i\theta^{a(+,0)}\bar\theta_a^{(-,0)} \ , \qquad \text{and} \qquad z^- = x^- - i\theta^{\bar a(0,+)}\bar\theta_{\bar a}^{(0,-)} \ . \tag{A.39}$$

In this basis, the supercharges become

$$\begin{aligned}
\mathcal{Q}_a^{(+,0)} &= -\frac{\partial}{\partial\theta^{a(-,0)}} + 2i\bar\theta_a^{(+,0)}\partial_+ \ , \qquad & \mathcal{Q}_a^{(-,0)} &= \frac{\partial}{\partial\theta^{a(+,0)}} \ , \\
\mathcal{Q}_{\bar a}^{(0,+)} &= -\frac{\partial}{\partial\theta^{\bar a(0,-)}} + 2i\bar\theta_{\bar a}^{(0,+)}\partial_- \ , \qquad & \mathcal{Q}_{\bar a}^{(0,-)} &= \frac{\partial}{\partial\theta^{\bar a(0,+)}} \ ,
\end{aligned} \tag{A.40}$$

while the spinor derivatives read

$$\begin{aligned}
\mathcal{D}_a^{(+,0)} &= -\frac{\partial}{\partial\theta^{a(-,0)}} \ , \qquad & \mathcal{D}_a^{(-,0)} &= \frac{\partial}{\partial\theta^{a(+,0)}} - 2i\bar\theta_a^{(-,0)}\partial_+ \ , \\
\mathcal{D}_{\bar a}^{(0,+)} &= -\frac{\partial}{\partial\theta^{\bar a(0,-)}} \ , \qquad & \mathcal{D}_{\bar a}^{(0,-)} &= \frac{\partial}{\partial\theta^{\bar a(0,+)}} - 2i\bar\theta_{\bar a}^{(0,-)}\partial_- \ .
\end{aligned} \tag{A.41}$$

Notice in particular that the operators $(\mathcal{D}_a^{(+,0)},\mathcal{D}_{\bar a}^{(0,+)})$ are simple partial derivatives with respect to the Grassmann variables.

A.3.2 Superfields

The superfields we are going to consider on (2.20) are of the following type[10] [24]

$$\begin{aligned}
\Phi^{(+,+)} = \ & \varphi^{(+,+)} + 2\theta^{a(+,0)}\psi_a^{(0,+)} + 2\theta^{\bar a(0,+)}\psi_{\bar a}^{(+,0)} - i\theta^{a(+,0)}\theta_a^{(+,0)}\partial_+\varphi^{(-,+)} \\
& - i\theta^{\bar a(0,+)}\theta_{\bar a}^{(0,+)}\partial_-\varphi^{(+,-)} + \theta^{a(+,0)}\theta^{\bar a(0,+)}F_{a\bar a} - 2i\theta^{a(+,0)}\theta^{\bar a(0,+)}\theta_{\bar a}^{(0,+)}\partial_-\psi_a^{(0,-)} \\
& - 2i\theta^{\bar a(0,+)}\theta^{a(+,0)}\theta_a^{(+,0)}\partial_+\psi_{\bar a}^{(-,0)} - \theta^{a(+,0)}\theta_a^{(+,0)}\theta^{\bar a(0,+)}\theta_{\bar a}^{(0,+)}\partial_+\partial_-\varphi^{(-,-)} \ .
\end{aligned} \tag{A.42}$$

Here we are using harmonically projected component fields, for example for the scalars

$$\varphi^{(\pm,\pm)} = \varphi^{i\bar\imath}u_i^{(\pm,0)}u_{\bar\imath}^{(0,\pm)} \ , \tag{A.43}$$

while the notation for the fermions shall be

$$\psi_{\bar a}^{(\pm,0)} = \psi_{\bar a}^{i}u_i^{(\pm,0)} \ , \qquad \text{and} \qquad \psi_a^{(0,\pm)} = \psi_a^{\bar\imath}u_{\bar\imath}^{(0,\pm)} \ . \tag{A.44}$$

Notice in particular that the superfield (A.42) has charges $(+1,+1)$ with respect to $U(1)_L \times U(1)_R$. Moreover, observe that the superfield $\Phi^{(+,+)}$ has an ultrashort component expansion, which is due to our formulation of the theory on harmonic superspace. To be more precise, $\Phi^{(+,+)}$ is G(rassmann)-analytic in the sense that it only depends on

[10] We are working in the analytic basis (A.39) and it is understood that all component fields are functions of $z^\pm$ as well as the harmonic coordinates.

half of all the Grassmann variables. Formulated as a constraint on $\Phi^{(+,+)}$ this statement means

$$\mathcal{D}_a^{(+,0)}\Phi^{(+,+)} = \mathcal{D}_{\bar{a}}^{(0,+)}\Phi^{(+,+)} = 0 \ , \tag{A.45}$$

which becomes apparent upon recalling (A.41). In a sense one can understand equation (A.45) as the generalised $\mathcal{N}=4$ version of the chirality conditions of $\mathcal{N}=2$ supersymmetry (*i.e.* equations (A.3), (A.8) and (A.11)). Besides that $\Phi^{(+,+)}$ is also H(armonically)-analytic, which means that the harmonic dependence of its components is not arbitrary but satisfies

$$D^{(+2,0)}\Phi^{(+,+)} = D^{(0,+2)}\Phi^{(+,+)} = 0 \ . \tag{A.46}$$

B The $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal theories

B.1 The $\mathcal{N}=2$ and $\mathcal{N}=4$ superconformal algebras

The $\mathcal{N}=2$ superconformal algebra is generated by the modes L_n of the stress-energy tensor, the modes J_n of a $U(1)$-current, as well as by the modes of the two supercurrents $G_r^{\pm}$. The commutation relations are

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{c}{12}(m^3-m)\delta_{m,-n} \ , \\
[L_m, J_n] &= -nJ_{m+n} \ , \\
\left[L_m, G_n^{\pm}\right] &= \left(\tfrac{1}{2}m - n\right) G_{m+n}^{\pm} \ , \\
[J_m, J_n] &= \tfrac{c}{3}m\delta_{m,-n} \ , \\
\left[J_m, G_n^{\pm}\right] &= \pm G_{m+n}^{\pm} \ , \\
\left\{G_m^+, G_n^-\right\} &= 2L_{m+n} + (m-n)J_{m+n} + \tfrac{c}{3}(m^2 - \tfrac{1}{4})\delta_{m,-n} \ , \\
\left\{G_m^+, G_n^+\right\} &= \left\{G_m^-, G_n^-\right\} = 0 \ .
\end{aligned}$$

Here the superscript $\pm$ denotes the $U(1)$ charge of the generator. In the $\mathcal{N}=4$ superconformal algebra the $U(1)$ symmetry is enhanced to an affine $SU(2)$ symmetry; the corresponding modes are denoted by $J_n^{\pm}$ and J_n, and satisfy the commutation relations

$$\begin{aligned}
[J_m, J_n^{\pm}] &= \pm 2\, J_{m+n}^{\pm} \\
[J_m^+, J_n^-] &= J_{m+n} + \frac{c}{6} m\, \delta_{m,-n} \\
[J_m, J_n] &= \frac{c}{3}\, m\, \delta_{m,-n} \, .
\end{aligned}$$

In addition there are two more supercurrents, whose modes we denote by $G'^{\pm}_r$. The additional commutation relations are [51]

$$\begin{aligned}
\{G_m^{\pm}, G'^{\pm}_n\} &= \mp 2(m-n)J_{m+n}^{\pm} & \{G_m^{\pm}, G'^{\mp}_n\} &= 0 \\
[L_m, G'^{\pm}_n] &= \left(\frac{m}{2} - n\right) G'^{\pm}_{m+n} & [J_m, G'^{\pm}_n] &= \pm G'^{\pm}_{m+n} \\
[J_m^{\pm}, G_n^{\pm}] &= [J_m^{\pm}, G'^{\pm}_n] = 0 & & \\
[J_m^{\pm}, G_n^{\mp}] &= \pm G'^{\pm}_{m+n} & [J_m^{\pm}, G'^{\mp}_n] &= \mp G_{m+n}^{\pm} \\
\{G'^{+}_m, G'^{-}_n\} &= 2L_{m+n} + (m-n)J_{m+n} + \frac{c}{3}(m^2 - \tfrac{1}{4})\delta_{m,-n} \, . & &
\end{aligned} \tag{B.1}$$

The relation of these operators to the supercharges of the superspace approach of appendix A.2 is

$$\mathcal{Q}^i = \begin{pmatrix} G'^{+}_{-1/2} \\ G^{+}_{-1/2} \end{pmatrix} , \qquad \text{and} \qquad \bar{\mathcal{Q}}^i = \begin{pmatrix} G^{-}_{-1/2} \\ -G'^{-}_{-1/2} \end{pmatrix} . \tag{B.2}$$

It is then straightforward to reproduce the action of $SU(2)_c$ as well as the relation (A.16)

$$\{\mathcal{Q}^i, \bar{\mathcal{Q}}_j\} = 2\delta^i_j L_{-1}, \tag{B.3}$$

upon the identification of $L_{-1} = -i\partial_+$ with the generator of infinitesimal bosonic translations.

B.2 Fusion rules for $\mathcal{N} = 2$

In this appendix we summarise the fusion rules of chiral and anti-chiral primary fields. In general, there are three different types of fusion rules for $\mathcal{N} = 2$ theories: the even fusion rules and the two odd fusion rules [39]. These are characterised by the property that either

$$\langle \varphi, \phi_1 \otimes \phi_2 \rangle \neq 0 \qquad \text{even} \tag{B.4}$$

or

$$\langle \varphi, \left(G^{\pm}_{-1/2}\phi_1 \right) \otimes \phi_2 \rangle \neq 0 \qquad \pm \text{ odd} . \tag{B.5}$$

Obviously, if ϕ_1 is a chiral primary field, only the even and $-$odd fusion rule can be non-trivial. This is also true if ϕ_2 is a chiral primary, as can be seen from symmetry, or by using (B.7) below.

First we want to show that in the even fusion rule only one field can appear, namely the chiral primary with $q = q_1 + q_2$. It is clear by charge conservation that φ must have $q = q_1 + q_2$; thus it suffices to show that φ must be chiral primary. This follows because

$$\begin{aligned} (2h-q)\langle \varphi, \phi_1 \otimes \phi_2 \rangle &= \langle G^{-}_{1/2} G^{+}_{-1/2}\varphi, \phi_1 \otimes \phi_2 \rangle \\ &= \langle G^{+}_{-1/2}\varphi, \Delta(G^{+}_{-1/2})(\phi_1 \otimes \phi_2) \rangle = 0 , \end{aligned} \tag{B.6}$$

where Δ denotes the comultiplication [40] which in this case takes on the simple form

$$\Delta(G^{+}_{-1/2}) = G^{+}_{-1/2} \otimes \mathbf{1} + \mathbf{1} \otimes G^{+}_{-1/2} . \tag{B.7}$$

This then vanishes, because both ϕ_1 and ϕ_2 are chiral primaries. In the case of interest, $h_1 = h_2 = \frac{1}{2}$ and $q_1 = q_2 = 1$, so that we have

$$\phi_c \otimes \phi_c = [\varphi_{h=1,q=2}] \oplus [G^{+}_{-1/2}\varphi^{+}_{q=1}] , \tag{B.8}$$

where φ^+ has $h^+ > \frac{1}{2}$, and $[\psi]$ is the $\mathcal{N} = 2$ representation generated from the state ψ.

B.3 Fusion rules for $\mathcal{N}=4$

Finally we want to collect some facts about the fusion rules of $\mathcal{N}=4$ representations.[11] We shall only consider the NS sector. Furthermore, we shall only discuss the case that is of interest in the current context, namely the fusion of two 'massless' representations with $j=\frac{1}{2}$ at $k=1$. The massless representations are those that saturate the BPS bound, which in this context means that they have $h=\frac{1}{2}$. The Virasoro highest weight states of the representation with $j=\frac{1}{2}$ therefore forms a doublet (ϕ_c,ϕ_a) with

$$\begin{array}{rclrcl} L_0\phi_c & = & \frac{1}{2}\phi_c & \qquad L_0\phi_a & = & \frac{1}{2}\phi_a \\ J_0\phi_c & = & \phi_c & \qquad J_0\phi_a & = & -\phi_a \\ J_0^+\phi_c & = & 0 & \qquad J_0^+\phi_a & = & \phi_c \\ J_0^-\phi_c & = & \phi_a & \qquad J_0^-\phi_a & = & 0\ . \end{array} \tag{B.9}$$

With respect to the usual $\mathcal{N}=2$ subalgebra, ϕ_c is thus a chiral primary state, while ϕ_a is an anti-chiral primary state. Note that the same is also true for the $\mathcal{N}=2$ subalgebra generated by $G'^{\pm}_r$, as was already noticed in [52].

To study the fusion rules of two such representations we first collect what we know based on the fusion rules with respect to the two $\mathcal{N}=2$ subalgebras (the ones generated by $G^{\pm}_r$ and $G'^{\pm}_r$, respectively). Using the same arguments as in the previous section, and combining the constraints coming from the two $\mathcal{N}=2$ algebras we know that the fusion rules of two chiral primary fields are

$$\phi_c\otimes\phi_c=[\varphi_{h=1,q=2}]\oplus[G^+_{-1/2}G'^+_{-1/2}\varphi_{h,q=0}]\ . \tag{B.10}$$

Here, the conformal families are written with respect to either of the $\mathcal{N}=2$ algebras. On the other hand, by the arguments of section 3.3 we also know that the fusion of ϕ_c and ϕ_a does not allow for any odd fusion rules, and thus is of the form

$$\phi_c\otimes\phi_a=[\tilde{\varphi}_{h,q=0}]\ . \tag{B.11}$$

Finally, we can use the constraints coming from the affine su(2) symmetry. At $k=1$ we know that the fusion of two $j=1/2$ representations only leads to a $j=0$ representation (since $j=1$ is not allowed at $k=1$). Furthermore, if $\tilde{\varphi}$ is any $\mathcal{N}=4$ highest weight state that appears in the fusion of ϕ_c with ϕ_a it then follows that

$$\begin{aligned} 0\neq\langle\tilde{\varphi}\,|\,\phi_a(1)\,\phi_c(0)\rangle &= \langle\tilde{\varphi}\,|\,(J_0^-\phi_c)(1)\,\phi_c(0)\rangle \\ &= \langle\tilde{\varphi}\,|\,J_1^-\Big(\phi_c(1)\,\phi_c(0)\Big)\rangle \\ &= \langle J^+_{-1}\tilde{\varphi}\,|\,\phi_c(1)\,\phi_c(0)\rangle\ . \end{aligned} \tag{B.12}$$

Here we have written out the explicit form of the scalar product and, in the second line, have made use the comultiplication formula. Thus for every $\tilde{\varphi}$ that appears on the right hand side of (B.11), $J^+_{-1}\tilde{\varphi}$ must appear on the right hand side of (B.10)! As we have argued before (based on the representation theory of the $\mathcal{N}=4$ algebra at $k=1$), the first term in (B.10) is of this form, since

$$\varphi_{h=1,q=2}=J^+_{-1}\Omega\ . \tag{B.13}$$

[11] As far as we are aware, this problem has not been addressed before in the literature.

The second type of terms in (B.10) is actually also of this form, since at $k=1$ we always have the null relation

$$\Big(J^{+}_{-1} - \frac{1}{2h} G^{+}_{-1/2} G'^{+}_{-1/2}\Big) \varphi_{h,q=0} = 0 \ , \tag{B.14}$$

where $\varphi_{h,q=0}$ is in the singlet representation of the su(2) zero modes, and we have assumed that $h>0$. This shows that we can write the above fusion rules more compactly as

$$\phi_c \otimes \phi_c = [J^{+}_{-1} \varphi_{h,q=0}] \ , \qquad \phi_c \otimes \phi_a = [\varphi_{h,q=0}] \ , \tag{B.15}$$

where $\varphi_{h,q=0}$ is in the singlet representation of $\mathcal{N}=4$ and both instances refer to the *same* representation. In particular, the representations on the right hand side then combine into one $\mathcal{N}=4$ representation, as must be the case (since the states on the left-hand sides also lie in the same $\mathcal{N}=4$ representations). Thus we can write more succinctly

$$[\phi_{h=1/2,j=1/2}] \otimes [\phi_{h=1/2,j=1/2}] = [\varphi_{h,j=0}] \,, \tag{B.16}$$

where now the conformal families refer to $\mathcal{N}=4$ families. Unfortunately, we have not been able to deduce any non-trivial constraints on the possible values of h that appear on the right hand side, and the example of section 4.2 suggests that no such general constraint exists.

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